

Programs involved in this report (3.1):

634 Mathematics: / Bachelor of Arts* (L&S) / Bachelor of Science* (L&S) / Master of Arts-Mathematics (L&S) / Doctor of Philosophy (L&S) / / 634 Certificate in Mathematics (L&S)

Contact (2.1)

Email (2.3)

Address (2.3)

Phone (2.4)

Names consistent with official lists (3.2)? yes

Is your plan current (4.1)? **Yes**

If plan is not current, plans to update? (4.2)

Do you have other programs on which you can provide a report (4.3)

Assessment Purpose (5.1)

Report on the Math Department's MIU project, / / and: / / Assessment of Math 421 /

Outcomes/Goals Assessed (5.2):

Math 421 is a course for math majors. The course is meant to teach math majors to read and write modern mathematics, and attempts to do so by exploring the theoretical underpinnings of single variable calculus. In particular, math 421 should prepare students for the higher level math courses, and for math 521 in particular. / / The (limited) assessment goal was to see if students who have completed math 421 appear adequately prepared for the upper level math courses. / /

Assessment Strategy (5.3):

Assessment was done by reading and classifying student responses to certain questions on the final exam.

Key Findings (5.4):

Student performance in math 421 seemed disappointingly low, and one would expect a large fraction of the students who finished math 421 do rather poorly in math 521. / It is not clear what caused this poor performance, although the very large class size (60 students for a course that is classified as "writing intensive") does stand out. / / This fall (2014) the Undergraduate Program Committee of the Mathematics Department will consider the available options for improving the situation. /

Nest Steps (6.1):

In fall 2014 we plan to assess how well the linear algebra courses Math 340 and Math 341 prepare students for abstract mathematical reasoning, by means of embedded questions in the final exam. /

ASSESSMENT OF MATH 421, SPRING 2013

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1. INTRODUCTION

The Undergraduate Program Committee of the Mathematics Department decided to assess the writing intensive course math 421 in spring 2013. It was also decided that the two Linear Algebra courses, Math 340 and Math 341, should be assessed in Fall 2014. Assessment of Math 421 was done through an analysis of student responses to questions on the final exam.

2. BACKGROUND

Undergraduate students at UW Madison can enter the math major after successfully completing the three course calculus sequence Math 221/222/234. These courses are primarily service courses which are intended to provide an introduction to the subject for students interested in the Sciences or Engineering. The calculus sequence emphasizes an intuitive understanding of the subject, and leads most students to spend their efforts on mastering the mechanical aspects of the computations involved in the calculus (hence the name “calculus.”)

The student who goes on to take more advanced math courses finds that the nature of the subject (and courses) changes: the emphasis on computation diminishes and is replaced by formal reasoning, often combined with an increased level of abstraction. The calculus courses do not prepare the average math major for the upper level math courses¹. To provide this missing preparation the math department offers a small number of bridge courses in which students are taught how to read, formulate, and write mathematical arguments. The three principal courses with this role are Math 341 (Linear Algebra), Math 371 (Basic Concepts

¹This is not the case in the honors calculus sequence math 275/276/375/376, where much greater emphasis is placed on mathematical reasoning. However, the honors students form a minority within the group of all math majors.

of Mathematics), and Math 421 (Theory of Single Variable Calculus), the course currently being assessed.

The subject of Math 421 consists of the proofs and deeper reasoning that underlie first semester calculus (single variable calculus). Principal topics are

- The least upper bound axiom characterizing completeness of the real numbers,
- Limits, Continuity, and Differentiability of Functions,
- The Intermediate Value Theorem, the Mean Value Theorem, and the fact that any continuous function on a closed interval attains a maximum,
- Riemann's definition of the integral in terms of upper and lower sums.

The course math 421 was created around the year 2000. Originally it was meant to be a writing intensive course in the sense of the description at <http://vanhise.lss.wisc.edu/wac/?q=node/105> ("Requirements for Writing-Intensive Courses," on the "Writing Across the Curriculum" web site). In particular, the course was intended to feature frequent homework assignments that would require students to produce coherent written mathematical arguments. Assignments were meant to be corrected and edited by the instructor, who was to give feedback on how students could improve their writing, and in particular, the logic in their arguments. Moreover, the homework assignments were supposed to lead to discussions between student and instructor on the validity and nature of mathematical reasoning.

The guidelines for writing intensive courses² recommend that enrollment in such courses should be kept lower than 30, and while it was initially easy to respect this bound, this has become much harder with the recent expansion of the math major and the resulting drastic increase in the enrollment in math 421. In

	'08	'09	'10	'11	'12	'13
spring	14	21	31	42	40	60
fall	15	26	35	49	47	?

Math 421 enrollment

this most recent spring semester enrollment in the course reached 60 students for one section. The course was taught by one lecturer and one MIU-funded teaching assistant. The lecturer, Dr. Gabriele Meyer, is a respected and experienced member of the math department, who, in the same semester, was also teaching one of the large math 222 lectures, as well as the course Math 371. She also serves as undergraduate advisor to math majors and math certificate students, and she is known for her work in the interface between Mathematics and Art.

For the last four years the course has been offered every semester. In those four years the course was taught by a post doc or short term visitor, except for the two occasions when Dr. Meyer was the instructor.

²<http://vanhise.lss.wisc.edu/wac/?q=node/105>
 "Requirements for Writing-Intensive Courses," on the "Writing Across the Curriculum" web site

Due to the ever-increasing class size and the transient nature of the pool of instructors for the course, the writing intensive practices have fallen into disuse. For the assessment we therefore considered student responses to final exam questions that Dr. Meyer gave as the main indicator for student learning in the course. The exam is included in the Appendix. It consists of seven questions, of which we singled out questions 1, 2, 3, and 6, for assessment.

3. WHO TAKES MATH 421?

Since the course is meant to be a first introduction to reading and writing mathematical proofs, math 421 is closed to students with credit from the honors calculus sequence. By design this excludes the mathematically more precocious students from the population that takes math 421.

While the course is intended for math majors, the following table of majors of the students enrolled in math 421 during spring 2013, shows that the course is also taken by students from many other majors.

Math Only Majors	Math Certificate
19 math	3 Econ & Math-cert
Dual Math Majors	1 Polisci, French, & Math-cert
4 math & Econ	Other
3 math & Stat	7 none
2 math & Econ & Stat	6 Econ
2 math & CS	2 Stat
2 math & Chinese	2 Computer Engineering
1 math & Chem & Econ	1 Engineering Mechanics
1 math & Ag.Econ	1 Econ & Stat
1 math & Business	1 Interior Design
16 total (dual majors)	1 History
35 total (all math majors)	25 total (certificates & other)

A number of the non-math majors turn out already to have so many math credits that they are close to completing a math major. One of them seems to be taking all the math courses that are required by modern Masters of Finance programs.

Math 421 (and 521) is one of the recommended courses for the mathematical option in the Statistics major.

Passing Math 421 (and 521) increases an Economics student's chances of entering Economics graduate school, which explains the motivation for the six Econ majors.

4. DISCUSSION OF THE EXAM QUESTIONS AND STUDENTS RESPONSES

For the assessment we read the students solutions to problems 1, 2, 3, and 6 on the final exam, and classified each student's response as valid, partially valid, or not valid. In general a valid response indicates that the student has achieved the

level of mastery in mathematical writing that will hopefully allow the student to do well in the 500 level math courses. The results are tabulated in the next section, but first we discuss the problems and the parts of the course material they test in more detail.

Problem 1 presents students with a continuous function that is defined on the whole real line, and that is its own inverse. The problem is to show that for such a function there always exists at least one real number x for which $f(x) = x$. One possible solution runs as follows

- (1) Argue by contradiction and assume that $f(x) \neq x$ for all x , then
- (2) Use continuity of the function f to conclude that either $f(x) > x$ for all x or $f(x) < x$ for all x ; realize one must separate the two cases where either $f(x) > x$ for all x , or alternatively, where $f(x) < x$ for all x .
- (3) Note that if for all x one has $f(x) > x$, then one can choose some x , say $x = 0$, and conclude $f(0) > 0$, and also $f(f(0)) > f(0)$;
- (4) The property that $f = f^{-1}$ implies that $f(f(x)) = x$ for all x , and in particular, $f(f(0)) = 0$. In the previous step we have thus found that both $f(0) > 0$ and $0 = f(f(0)) < 0$, a contradiction.
- (5) End by stating that the other case, in which $f(x) < x$ for all x , leads to a similar contradiction by following very similar arguments.

Of the problems on the exam, this was probably the most intricate. A properly written solution along these lines involves argument by contradiction, use of quantifiers (“for all” and “there is”), separating an argument into distinct sub-cases, and knowledge of the Intermediate Value Theorem. On the other hand, students in the 500 level math courses will be expected to be somewhat comfortable with these modes of reasoning.

We found that of the 59 students who took the exam, only eight came up with a valid solution, while four came up with a partially valid solution. We considered a solution valid if most of the reasoning (logical connectives, use of the hypotheses) was included; a solution was considered partially valid if some fragments of a reasoning that could have led to a complete solution were included. The remaining cases were considered “no solution.” As with all problems in this assessment, grammar or punctuation were not taken into account.

Some students found other valid arguments, e.g. by using the hypothesis that f is continuous and invertible to conclude that f is either strictly increasing or strictly decreasing. One student provided a geometric argument (namely, if $f = f^{-1}$, then the graph of f must be invariant under reflection in the diagonal $y = x$; hence its graph must either be contained in the diagonal, or else contain points both above and below the diagonal; since f is continuous its graph must therefore “cross the diagonal,” or coincide with the diagonal). These arguments were counted among the valid solutions.

Problem 2 asks the student to show that the polynomial $f_m(x) = x^3 - 3x + m$ cannot have two or more roots in the interval $[0, 1]$, for any value of the constant m .

Within the context of Math 421 this is a question about either the Mean Value Theorem or Rolle's theorem. There is however also a more intuitive solution that is accessible to students from math 221. Namely, by computing the derivative and observing that $f'_m(x) < 0$ for $0 < x < 1$ one shows that f_m is decreasing on $[0, 1]$, so that there cannot be two values of $x \in [0, 1]$ with $f_m(x) = 0$. Of course, a central point of math 421 is that one needs the Mean Value Theorem to prove the fact that a negative derivative implies that a function is decreasing. However, the question as phrased on the exam admits the shorter math 221 style solution and therefore, when classifying the solutions into valid or not valid, we counted the simple observation that $f'_m(x) < 0$ for $0 < x < 1$ precludes two roots in the interval $[0, 1]$ as valid.

Given this broad interpretation of a “valid solution,” one out of three students could not come up with a valid solution.

A small number of students made basic algebra mistakes (such as “ $x(x^2 - 3) = m \Rightarrow x = m$ or $x^2 - 3 = m$ ”).

Problem 3 asks the students to prove that a number $x \in [a, b]$ exists such that $\int_a^x f = \int_x^b f$, where f is a Riemann integrable function on the interval. The student is also asked to provide an example where x must be one of the endpoints of the interval.

One possible solution is to consider the function representing the difference between the two integrals, i.e. $g(x) = \int_a^x f - \int_x^b f$, and prove that it must vanish somewhere in the interval $[a, b]$. The most straightforward way to guarantee existence of a solution to $g(x) = 0$ is to use the Intermediate Value Theorem and check the signs of $g(a)$ and $g(b)$. One finds

$$g(a) = - \int_a^b f, \quad g(b) = + \int_a^b f.$$

Hence either $g(a) = 0$, and we have a solution ($x = a$), or $g(a)$ and $g(b) = -g(a)$ have opposite signs, so that the Intermediate Value Theorem implies existence of a solution to $g(x) = 0$ with $a < x < b$.

The three important ingredients in this solution are:

- the idea to consider the function $g(x)$,
- the fact that the antiderivative of an integrable function always is continuous, so that g is continuous, and
- the Intermediate Value Theorem.

It follows from the proof that to find an example of a function f for which the solution x must be one of the endpoints, one must choose a function f with $\int_a^b f = 0$. E.g. the choice $f(x) = \sin x$ on the interval $[0, 2\pi]$ leads to $g(x) = 2 - 2 \cos x$, for which $g(x) > 0$ for all x with $0 < x < 2\pi$.

Many students gave a geometric argument, in which they interpret both sides in the equation $\int_a^x f = \int_x^b f$ as areas of regions “under the graph” of f , and observed that as one moves x from a to b , there must be a point at which both areas are equal. This argument has some virtue although it does ignore the possibility

that the function f might change sign, and that the function f could be very complicated (changing sign infinitely often, etc).

In the assessment of this problem, a write-up containing the main ingredients of the solution given above, or else a clear exposition of the geometric argument, were both counted as valid. A solution in which one or two parts of a valid solution were present, and which did not contain further contradictions, was counted as partially valid. Out of 59 students, 14 wrote valid solutions, 15 wrote partial solutions, and 30 students (half the class) wrote non-valid solutions.

Problem 6 is a straightforward question about Riemann's upper and lower sum definition of integrability. The student is asked to show that a simple step function is integrable. The student is not asked to compute the integral (which is trivial, being the area of a unit square).

A straightforward and elementary proof begins with assuming $\varepsilon > 0$ is arbitrarily given and then exhibiting a partition of the interval $[0, 2]$ for which the upper and lower sums differ less than ε . The simplest partition with this property is $\mathcal{P} = \{0, 1 - \varepsilon, 1, 2\}$. Any other partition $\{0 = x_0, x_1, \dots, x_n = 2\}$ for which the interval (x_{k-1}, x_k) containing the number 1 has length at most ε will also do. Of the 59 students taking the exam only two or three found the simplest partition. The other students who found a valid solution considered an arbitrary partition and discovered the condition $x_k - x_{k-1} < \varepsilon$ mentioned before.

Many other students came up with partially valid solutions, which did not go back to the definition, but instead used theorems to the effect that any continuous function on a closed interval is integrable, or that if a function is integrable on $[a, b]$ and on $[b, c]$, then the function is integrable on $[a, c]$. Not bothered by the overkill involved in this argument, many of the students who chose this path also ignored the fact that the given function is not continuous on the closed interval $[0, 1]$.

Half the class could not produce anything like a partial solution to this problem. They clearly did not understand Riemann's definition of the integral.

5. CONCLUSION

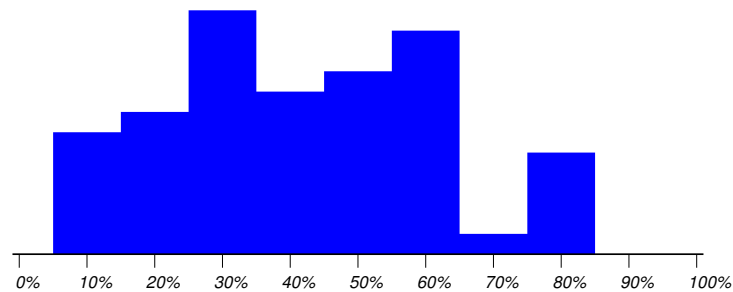
The following table summarizes the results of questions 1,2,3, and 6 of the math 421 final exam. The bar graph on the next page presents the final exam scores given by Dr. Meyer and her TA.

Problem	no solution	partial solution	valid solution
1	47	4	8
3	37	11	11
6	30	15	14
2	22	–	37

[Total number of students who took the final exam: 59]

Final exam scores, Math 421, May 2013

mean = 41% sdev = 19% median = 40% #students = 59



It is clear that the students' performance in this class is poor by any standard. It is more difficult to use a one-time assessment such as this one to get at the cause of this poor performance. Nevertheless, in view of the central role that this course has come to play in the math major, it is worth observing that

- (1) The students taking math 421 are those who did not come to mathematics through the honors program, and who probably made the decision to pursue mathematics rather late in their studies. They are therefore the students who are most in need of personal attention and generous feedback.
- (2) The writing intensive aspect of the course has been neglected over the last few years, perhaps as a result of the steady increase in class size. If the recommendations of the writing center were implemented then this single lecture of math 421 (with a TA) should really have been split in two lectures of 30 students, perhaps without TA. Lack of resources prevents the department from doing so.
- (3) With the exception of Dr. Meyer, the instructors who have taught the course over the last few years have all been temporary members of the math department, some of them spending as little as one semester in Madison. It would be good if there were a small group of faculty or highly qualified academic staff (such as Dr. Meyer) who would take ownership of the course and teach it on a regular basis (e.g. in the way that the probability group takes care of math 431, 632, and other probability courses). Again, lack of resources will keep this from happening.

6. APPENDIX: THE MATH 421 FINAL EXAM OF SPRING '13

- (1) If f is a continuous function on \mathbb{R} and $f = f^{-1}$, prove that there is at least one x such that $f(x) = x$.
- (2) Prove that the polynomial $f_m(x) = x^3 - 3x + m$ never has two roots in $[0, 1]$, for all $m \in \mathbb{R}$.
- (3) Suppose that f is integrable on $[a, b]$. Prove that there is a number x in $[a, b]$, such that

$$\int_a^x f = \int_x^b f.$$

Show by example that it is not always possible to choose x in (a, b) .

- (4) Suppose that f is continuous on $[a, b]$. Prove that for some $\xi \in [a, b]$

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx.$$

- (5) Find a function g such that $\int_0^{x^2} tg(t)dt = x + x^2$.
- (6) Suppose $f : [0, 2] \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Show that f is integrable.

- (7) Prove that if $f''(a)$ exists, then

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

The limit on the right is called the Schwarz second derivative of f at a .

Hint: Use the degree 2 Taylor polynomial with $x = a + h$ and $x = a - h$.

Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = -x^2$ for $x \leq 0$. Show that

$$f''(0) = \lim_{h \rightarrow 0} \frac{f(0+h) + f(0-h) - 2f(0)}{h^2}$$

exists, even though $f''(0)$ does not.

7. APPENDIX: REQUIREMENTS FOR WRITING INTENSIVE COURSES

The Writing Across the Curriculum site lists the minimum requirements for a course to be writing intensive” at <http://vanhise.lss.wisc.edu/wac/?q=node/105>.

- (1) Writing assignments must be an integral, ongoing part of the course, and the writing assignments must constitute a substantial and clearly understood component of the final course grade. Assignments must be structured and sequenced in such a way as to help students improve their writing. Instructors in writing-intensive courses should not just assign writing; they should help students succeed with and learn from that writing.
- (2) There must be at least four discrete writing assignments spread throughout the semester, not including in-class essay exams.
- (3) At least one assignment must involve revision; the draft and revision may count as two discrete writing assignments. Exceptions will be allowed for instructors who instead choose to use a sequence of repeated assignments.
- (4) Students must produce a total of at least 14 double-spaced pages (c. 4000 words) of finished prose; this total does not include pages in drafts. When the writing is in a foreign language, a lower number of total pages may be appropriate.
- (5) Instructors must provide feedback on students’ writing assignments.
- (6) Some class time must be devoted to preparing students to complete writing assignments. Some options include:
 - discussion of assignments and of evaluation criteria
 - analysis and discussion of sample student papers
 - discussion of writing in progress, using examples of successful work from students
 - peer group activities that prepare students to write a particular paper, such as sharing and discussion of plans, outlines, strategies, theses, drafts
 - discussion or presentations of students’ research in progress
 - instruction about how to write a particular type of paper or about solving a common writing problem